A Three-Dimensional Modified Strongly Implicit Procedure for Heat Conduction

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The application of discretization techniques frequently leads to a system of algebraic equations having a well-defined coefficient structure. A modified strongly implicit procedure for solving the system resulting from the modeling of heat conduction in three dimensions is presented in this work. The method is derived for a 19 point scheme with the more common 7 point scheme emerging as a special case of the procedure. In this way, the asymmetric influence of the additional terms in the LU matrix product is weakened. As a consequence, the method is less sensitive to the iteration parameter and mesh aspect ratio and, in addition, provides considerably more rapid convergence than does the strongly implicit procedure. The increased convergence is exhibited by a significant reduction in the computational cost. The characteristics of the method are examined through application to several model problems and application is made to a more complex three-dimensional problem. Comparisons with the SIP (strongly implicit) and ADI (alternating direction implicit) methods are provided.

Nomenclature

a,b,c,d,e,f,g	= coefficients in L
A,A'	= coefficient matrices
B	= additional matrix
D	= Taylor series expansion, defined in text
$\bar{E}_a$	= average error in domain
h,p,r,s	= coefficients in U
i,j,k	= location indices for the control volume
k_x,k_y,k_z	= thermal conductivity in the x, y, and z directions, respectively
L	= lower triangular matrix
P	= strength of heat source per unit volume
q	= right-side vector of finite difference equations
Q	= total heat flow rate
R	= residual vector
T	= temperature
t_f	= CPU time in seconds using FORTRAN G compiler on an IBM 370/158 installation
u	= coefficient in U
U	= upper triangular matrix
v	= coefficient in U
V	= intermediate vector
x,y,z	= Cartesian coordinates
α	= iterative parameter
β	= summation index
δ	= change in temperature for one iteration
Δ	= difference in accompanying variable
$\phi^1,\phi^2,\ldots\phi^{24}$	= coefficients in LU matrix product
ω	= relaxation parameter

Subscripts

i	= x location in grid
j	= y location in grid
k	= z location in grid

Superscripts

b	= back
e	= east
f	= front
n	= north, iteration level
p	= pole
s	= south
w	= west

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Introduction

In continuum problems, the application of conventional finite difference techniques to the analysis of conduction heat-transfer problems frequently leads to a system of algebraic equations having a well-defined structure. In three-dimensions, the common finite difference procedures give rise to a seven-point scheme and, in general, the system of equations can be represented by the matrix equation

$$[A]\{T\}=\{q\} \tag{1}$$

where the vector $\{T\}$ represents the temperature field at discrete locations within the domain.

While the simplest methods of solution for the above matrix equation are direct methods, these methods fail to take advantage of the well-defined structure of the coefficient matrix. Although direct methods are suitable for small systems, the cost of obtaining a solution to large sets of equations rapidly becomes prohibitive as the number of equations increases. This is particularly important in three-dimensional problems where even a relatively coarse mesh structure leads to a large number of equations. This realization has provided the major motivation for the development of iterative procedures that recognize the well-defined structure of the coefficient matrix.

One of the more common iterative procedures is successive over-relaxation (SOR), in which an initial guess field is successively improved through application of the equation for each discrete location. Through the application of under- or over-relaxation, the convergence of the guess field towards its final solution can be enhanced above that available through a simple application of the equation for each discrete location in the field. As the problem size increases, however, through finer mesh subdivisions, the convergence of such iterative procedures decreases as the large number of iterations increases, the cost of-solution also increases, and large systems of equations thereby become excessively expensive to solve.

In attempting to alleviate the slow convergence characteristics of the SOR procedures, the alternating direction implicit (ADI) procedure was proposed by Peaceman and Rachford.[1] In this procedure, the system of equations is successively rearranged so that at each stage of the process a tridiagonal system can be solved efficiently by direct means, using a tridiagonal matrix algorithm for the modified set of equations. Through appropriate selection of the rearrangement procedure, "line-by-line" solutions are obtained in each of the coordinate directions. This procedure has the advantage that the influence of boundary conditions is propagated throughout the entire domain when a complete iteration has been completed, where a complete iteration requires obtaining a line-by-line solution for each of the coordinate directions involved in the problem. The ADI method, being more implicit than the SOR method, generally offers a higher convergence rate than does the SOR method. However, both methods lose their effectiveness when complex problems are encountered and when the equation set becomes large.

More recently, an iterative, strongly implicit procedure (SIP) was proposed by Stone.[2] In this method the iterative procedure involves the direct, simultaneous solution of a set of equations formed by modification of the original matrix equation. The modified matrix is constructed according to two criteria: 1) the equation set must remain more strongly implicit than in the ADI case, and 2) the elimination procedure for the modified equation set must be economically efficient. The modified matrix equation of the SIP procedure has the form

$$[A+B]\{T\}=\{b\} \tag{2}$$

where

$$\{b\}\equiv\{q\}+[B]\{T\} \tag{3}$$

The form of $[B]$ is such that $\|[B]\|\ll\|[A]\|$ and that the decomposition of $[A+B]$ into a lower and upper triangular matrix product involves much less computation than the direct decomposition of $[A]$.

Since the right-hand side of Eq. (2) involves the unknown solution vector $\{T\}$, iteration is still required. The procedure given by Stone[2] is

$$[A+B]\{T\}^{n+1}=[A+B]\{T\}^{n}-\omega([A]\{T\}^{n}-\{q\}) \tag{4}$$

where a value of unity for ω was used. Although the SIP procedure can lead to a reduction in computational cost for certain problems, there remain several disadvantages to the procedure as proposed by Stone.[2] These are

1) The method is restricted to five-point schemes and two dimensions.

2) Reordering of the equations is required at each step of the iteration.

3) The rate of convergence is sensitive to the control volume or grid aspect ratio.

4) Convergence is highly problem dependent and in some cases is slower than that provided by ADI methods.

5) The $[L]$ $[U]$ matrix product is strongly asymmetric.

Dupont et al.[3] and Bracha-Barak and Saylor[4] adopted an approach similar to that proposed by Stone[2] and succeeded in obtaining a symmetric $[L]$ $[U]$ decomposed matrix product. Although they do obtain a symmetric $[L]$ $[U]$ decomposed matrix, equation reordering during the iteration process is still required and they do not discuss the sensitivity of their procedure to control volume aspect ratio or to the complexity of the problem being examined. Saylor[5] proposed a second-order symmetric factorization in which matrices $[A]$ and $[B]$ are both symmetric. His results indicate, however, that this method is unstable and therefore is not of practical utility. Schneider and Zedan[6] recently examined the application of strongly implicit procedures to two-dimensional field problems. They proposed a modified strongly implicit procedure (MSI) which was demonstrated to considerably reduce the computational expenditures required for solution, the complexity of application of the procedure, and the sensitivity of the procedure to both procedural and problem parameters. In addition, the modified strongly implicit procedure extended the applicability of the method to include application to nine-point formulations as is required in the finite difference method[7] and when use is made of nonorthogonal coordinate systems.[8]

Weinstein et al.[9] extended the SIP method to solve three-dimensional elliptic and parabolic differential equations. The same approach as was applied for two dimensions was utilized and their results indicated that the three-dimensional application not only retains the two-dimensional disadvantages but also is more sensitive to aspect ratio and iteration parameters. Further, they introduced three separate iterative parameters, one to be applied for each direction in space. To determine the optimal combination of these parameters is very expensive since the optimal combination depends on the characteristics of the particular problem, the number of control volumes, and the grid aspect ratio.

The motivation for the present work is therefore fourfold:

1) To extend the applicability of the method, without its disadvantageous features, to three-dimensional problems.

2) To reduce the computational effect required to obtain a converged solution for three-dimensional problems.

3) To remove the asymmetry of the $[L]$ $[U]$ product or to weaken its influence.

4) To reduce the sensitivity of the procedure to grid aspect ratio.

In the remainder of the paper, the modified strongly implicit procedure (MSI) will be developed for three-dimensional formulations. Following the development of the method, the sensitivity of the method to different parameters will be investigated. Comparisons with the SIP and ADI methods will be provided for several different problems.

Problem Formulation

The governing differential equation for three-dimensional steady-state heat conduction by Cartesian coordinates is given by

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)=-P \tag{5}$$

where P is the volumetric strength of the local heat source. By simply adopting a Taylor series approximation for the governing differential equation [Eq. (5)] or by performing an energy balance for the general control volume, as shown in Fig. 1, an algebraic equation can be formed for each control volume of the solution domain. Using approximations to Fourier's law to relate the heat flow shown in Fig. 1 to the nodal temperatures, the algebraic equation representing an energy balance on the control volume of Fig. 1 can be written in the form[10]

$$A^b_{i,j,k}T_{i,j,k-1}+A^s_{i,j,k}T_{i,j-1,k}+A^w_{i,j,k}T_{i-1,j,k}+A^p_{i,j,k}T_{i,j,k}$$
$$+A^e_{i,j,k}T_{i+1,j,k}+A^n_{i,j,k}T_{i,j+1,k}+A^f_{i,j,k}T_{i,j,k+1}=q_{i,j,k} \tag{6}$$

where $q_{i,j,k}$ represents the total heat source within the control volume centered at (i,j,k). The collection of equations in the form of Eq. (6), written for each location in the domain, together with boundary condition equations, which can also be expressed in the form of Eq. (6), yields the system of equations which requires solution. The structure of the resulting coefficient matrix is illustrated in Fig. 2. It is noted that the subscript notation (i,j,k) in the figure refers to the location within the grid network and that there are, in general, seven nonzero coefficients per row.

Solution Procedure for 19-Point Formulation

Weinstein et al.[9] modified the original matrix A by adding a small matrix B such that $(A+B)$ is factorable into L and U matrices which are sparse, and in which the nonzero elements have the same location as the corresponding elements in the original matrix. When the LU matrix multiplication is performed a modified matrix A' results. The modified matrix is

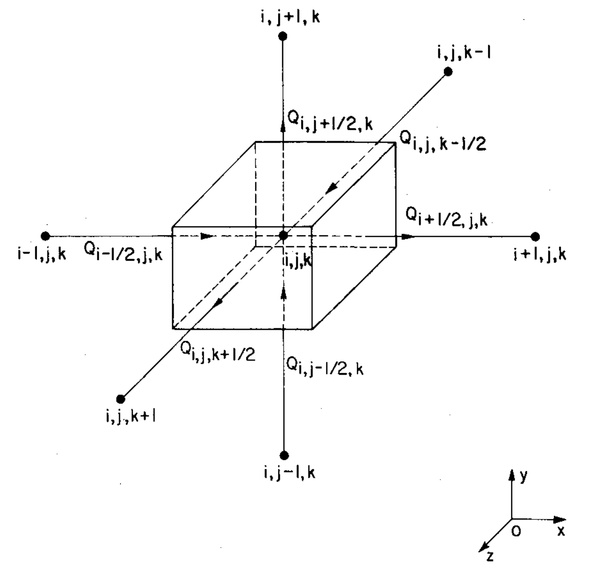


Fig. 1 General control volume.

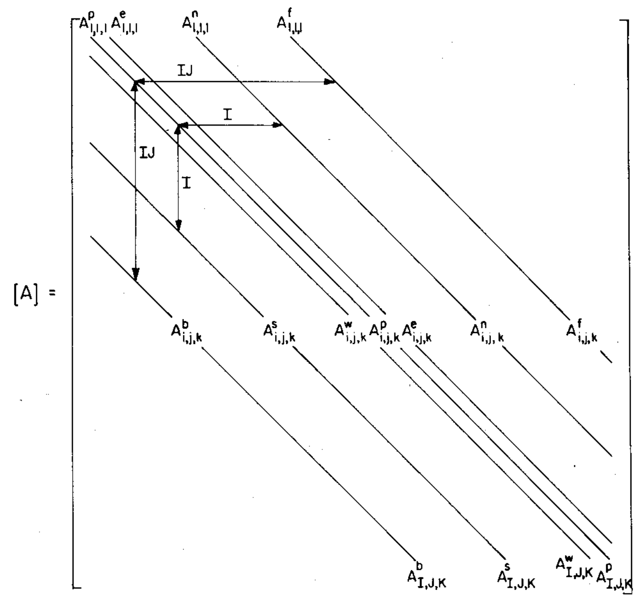


Fig. 2 Coefficient matrix for 7-point scheme.

given by

$$[A'] = [L][U] = [A+B] \tag{7}$$

In addition to the nonzero coefficients in the original matrix, there are six other nonzero coefficients. Since the nonzero entries in the original matrix A and the corresponding entries in the modified matrix A' are forced to be identical, the coefficients of L and U may be determined from the defining equations for the elements of A' by virtue of the LU product.

The most obvious extension of the MSI procedure to three dimensions is the application to a 27-point scheme. This approach, however, results in 36 additional terms. Due to the large number of additional terms involved in the 27-point scheme, it was decided to seek another appropriate scheme. The 19-point formulation, which has a 9-point molecule in each principle plane, is an appropriate alternative. The 7-point scheme can then be deduced as a special case of the 19-point scheme.

The finite difference representation of the governing differential equation [Eq. (5)] in the form of a 19-point formulation, can be derived by applying an energy balance to

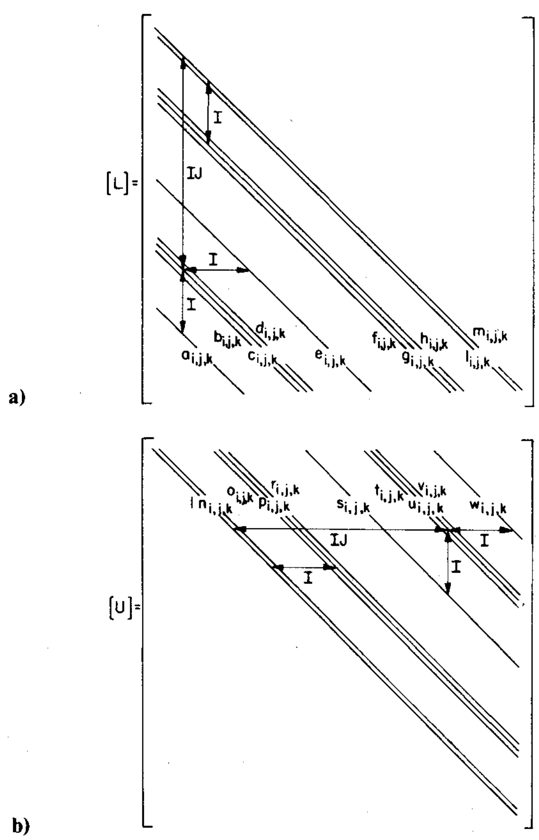


Fig. 3 L and U matrices for 19-point scheme.

a general control volume. To obtain the 19-point formulation, the assumption is made that the temperature gradient on the surfaces of the control volume can be represented by a polynomial of second order. Details are available in Ref. 11. The equation for a general point within the domain takes the form,

$$\begin{aligned}
&A^{bs}_{i,j,k} T_{i,j-1,k-1} + A^{bw}_{i,j,k} T_{i-1,j,k-1} + A^{b}_{i,j,k} T_{i,j,k-1} \\
&+ A^{be}_{i,j,k} T_{i+1,j,k-1} + A^{bn}_{i,j,k} T_{i,j+1,k-1} + A^{sw}_{i,j,k} T_{i-1,j-1,k} \\
&+ A^{s}_{i,j,k} T_{i,j-1,k} + A^{se}_{i,j,k} T_{i+1,j-1,k} + A^{w}_{i,j,k} T_{i-1,j,k} \\
&+ A^{p}_{i,j,k} T_{i,j,k} + A^{e}_{i,j,k} T_{i+1,j,k} + A^{nw}_{i,j,k} T_{i-1,j+1,k} \\
&+ A^{n}_{i,j,k} T_{i,j+1,k} + A^{ne}_{i,j,k} T_{i+1,j+1,k} + A^{fs}_{i,j,k} T_{i,j-1,k+1} \\
&+ A^{fw}_{i,j,k} T_{i-1,j,k+1} + A^{f}_{i,j,k} T_{i,j,k-1} + A^{fe}_{i,j,k} T_{i+1,j,k+1} \\
&+ A^{fn}_{i,j,k} T_{i,j+1,k+1} = q_{i,j,k}
\end{aligned} \tag{8}$$

The collection of equations of the form given by Eq. (8), including boundary condition equations, yields the matrix equation given as Eq. (1).

The L and U matrices are selected to have the form illustrated in Figs. 3a and 3b, respectively, such that each matrix has 10 nonzero elements per row, in the same locations as those of the original matrix A. The modified matrix A', as illustrated in Fig. 4, results from the LU matrix multiplication. It can be seen that, in addition to the 19 nonzero entries in the original matrix A, there are 24 additional entries, denoted as $\phi^{1}_{i,j,k}, \phi^{2}_{i,j,k}, \ldots \phi^{24}_{i,j,k}$. The equations to be used to determine the coefficients of L and U are the requirements that the original 19 coefficients in A remain unchanged in A'. The additional terms are then accepted as the value resulting from the LU product. The equations thereby formed can be solved, manually, to determine the elements of the L and U matrices and the resulting ϕ values. These extensive results are available in detail in Ref. 11.

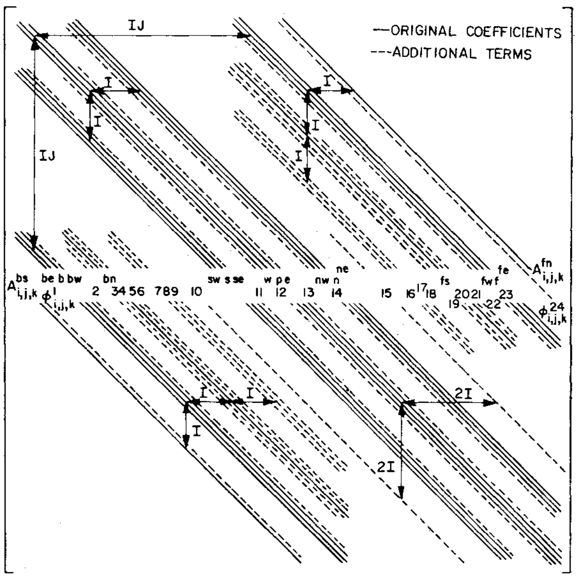


Fig. 4 *LU* **product for 19-point scheme.**

In order to increase the rate of convergence, partial cancellation of the influence of the additional terms is effected through the use of Taylor series expansions. The temperature at locations corresponding to the additional terms are expressed in terms of those temperatures involved in a 7-point scheme. The use of the termperatures involved in the 7-point scheme is made so that the more primitive 7-point scheme can always be employed as a special case of the 19-point scheme. Selected relations of the many that are actually required are given by

$$T_{i,j+2,k} = -T_{i,j,k} + 2T_{i,j+1,k} \tag{9}$$

$$T_{i+2,j,k-1} = -2T_{i,j,k} + 2T_{i+1,j,k} + T_{i,j,k-1} \tag{10}$$

$$T_{i+1,j+1,k-1} = -2T_{i,j,k} + T_{i+1,j,k} + T_{i,j+1,k} + T_{i,j,k-1} \tag{11}$$

$$T_{i-1,j+2,k-1} = -3T_{i,j,k} + 2T_{i,j+1,k} + T_{i-1,j,k} + T_{i,j,k-1} \tag{12}$$

$$T_{i+2,j+1,k-1} = -3T_{i,j,k} + 2T_{i+1,j,k} + T_{i,j+1,k} + T_{i,j,k-1} \tag{13}$$

A single equation in the modified set can be written in a general fashion in the form

$$\sum_{i=1}^{19} A_i T_i + \sum_{j=1}^{24} \phi^j T_j = q \tag{14}$$

where the first sum represents the original equation components and the second sum represents the additional terms. To implement the partial cancellation, and denoting by D_j the *appropriate* Taylor series expansion for a temperature T_j in terms of those involved in the original 7-point scheme, the above equation is rewritten in the form

$$\sum_{i=1}^{19} A_i T_i + \sum_{j=1}^{24} \phi^j (T_j - \alpha D_j) = q \tag{15}$$

where α is an iterative parameter introduced to permit some flexibility in optimizing convergence. By introducing this partial cancellation, the net influence of the additional terms can be reduced. Equation (15) is therefore more representative of the original equation requiring solution than is Eq. (14). Note, however, that the D_j terms contain temperatures which are also involved in the first summation on the left-hand side of Eq. (15). A regrouping of terms in the equation is therefore effected prior to determining the *LU* coefficients. Written out explicitly, the equation after this regrouping is written as:

$$\begin{aligned} & A^{bs}_{i,j,k} T_{i,j-1,k-1} + A^{bw}_{i,j,k} T_{i-1,j,k-1} + \bar{A}^{b}_{i,j,k} T_{i,j,k-1} \\ & + A^{be}_{i,j,k} T_{i+1,j,k-1} + A^{bn}_{i,j,k} T_{i,j+1,k-1} + A^{sw}_{i,j,k} T_{i-1,j-1,k} \\ & + \bar{A}^{s}_{i,j,k} T_{i,j-1,k} + A^{se}_{i,j,k} T_{i+1,j-1,k} + \bar{A}^{w}_{i,j,k} T_{i-1,j,k} \\ & + \bar{A}^{p}_{i,j,k} T_{i,j,k} + \bar{A}^{e}_{i,j,k} T_{i+1,j,k} + A^{nw}_{i,j,k} T_{i-1,j+1,k} \\ & + \bar{A}^{n}_{i,j,k} T_{i,j+1,k} + A^{ne}_{i,j,k} T_{i+1,j+1,k} + A^{fs}_{i,j,k} T_{i,j-1,k+1} \\ & + A^{fw}_{i,j,k} T_{i-1,j,k+1} + \bar{A}^{f}_{i,j,k} T_{i,j,k+1} + A^{fe}_{i,j,k} T_{i+1,j,k+1} \\ & + A^{fn}_{i,j,k} T_{i,j+1,k+1} + \sum_{\lambda=1}^{24} (\phi T)_\lambda = q_{i,j,k} \end{aligned} \tag{16}$$

where the superbars indicate modified coefficients which are defined below,

$$\bar{A}^{b}_{i,j,k} = A^{b}_{i,j,k} - \alpha \sum_{\beta=1}^{9} \phi^{\beta}_{i,j,k} \tag{17}$$

$$\begin{aligned} \bar{A}^{s}_{i,j,k} = A^{s}_{i,j,k} - \alpha \Big[& \phi^{1}_{i,j,k} + 2\phi^{10}_{i,j,k} + \phi^{11}_{i,j,k} \\ & + 2\sum_{\beta=16}^{18} \phi^{\beta}_{i,j,k} + \sum_{\beta=19}^{22} \phi^{\beta}_{i,j,k} \Big] \end{aligned} \tag{18}$$

$$\begin{aligned} \bar{A}^{w}_{i,j,k} = A^{w}_{i,j,k} - \alpha \Big[& \sum_{\beta=3,19,23} (2\phi^{\beta}_{i,j,k} + \phi^{\beta+1}_{i,j,k}) \\ & + 2\sum_{\beta=12,14} \phi^{\beta}_{i,j,k} + \sum_{\beta=7,16} \phi^{\beta}_{i,j,k} \Big] \end{aligned} \tag{19}$$

$$\begin{aligned} \bar{A}^{e}_{i,j,k} = A^{e}_{i,j,k} - \alpha \Big[& \sum_{\beta=1,5,21} (\phi^{\beta}_{i,j,k} + 2\phi^{\beta+1}_{i,j,k}) \\ & + 2\sum_{\beta=11,13} \phi^{\beta}_{i,j,k} + \sum_{\beta=9,18} \phi^{\beta}_{i,j,k} \Big] \end{aligned} \tag{20}$$

$$\begin{aligned} \bar{A}^{n}_{i,j,k} = A^{n}_{i,j,k} - \alpha \Big[& \sum_{\beta=3}^{6} \phi^{\beta}_{i,j,k} + 2\sum_{\beta=7}^{9} \phi^{\beta}_{i,j,k} \\ & + \phi^{14}_{i,j,k} + 2\phi^{15}_{i,j,k} + \phi^{24}_{i,j,k} \Big] \end{aligned} \tag{21}$$

$$\bar{A}^{f}_{i,j,k} = A^{f}_{i,j,k} - \alpha \sum_{\beta=16}^{24} \phi^{\beta}_{i,j,k} \tag{22}$$

$$\bar{A}^{p}_{i,j,k} = A^{p}_{i,j,k} + \alpha \Big[\sum_{\beta=1,4,7,\ldots}^{24} (\lambda_1 \phi^{\beta}_{i,j,k} + 2\phi^{\beta+1}_{i,j,k} + \lambda_2 \phi^{\beta+2}_{i,j,k}) \Big] \tag{23}$$

and where for

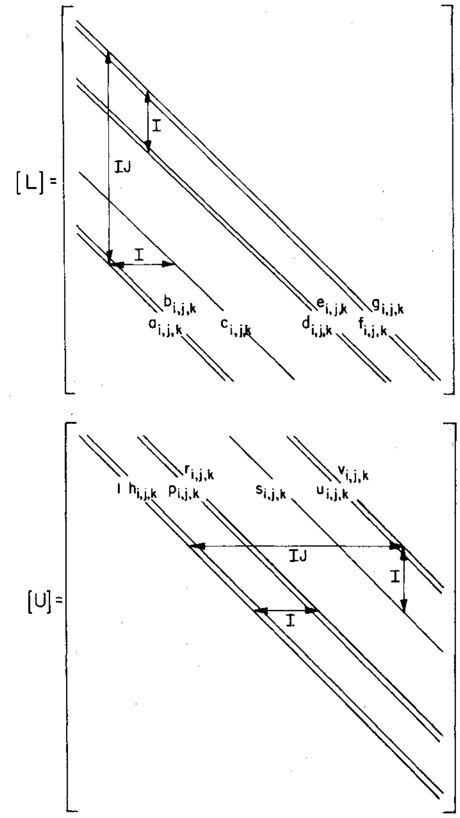
$$\beta = 1,4 \qquad \lambda_1 = 2,\ \lambda_2 = 3 \tag{24}$$

$$\beta = 7,16 \qquad \lambda_1 = \lambda_2 = 3 \tag{25}$$

$$\beta = 10,13 \qquad \lambda_1 = \lambda_2 = 1 \tag{26}$$

$$\beta = 19,22 \qquad \lambda_1 = 3,\ \lambda_2 = 2 \tag{27}$$

For purposes of calculating the *LU* matrix coefficients, the equations for their determination can be used simply by replacing the original coefficients by the superbarred coefficients where applicable. In essence, the introduction of this partial cancellation corresponds to altering the original conservation equation such that, when the *LU* product is formed, the partial cancellation is *inherent* in the modified matrix A'. It is also worthy of note that the use of the Taylor



a)

b)

Fig. 5 ***L* and *U* matrices for 19-point scheme.**

series representation and of the partial cancellation affects only the approach to convergence of the iterative algorithm and that these influences vanish when the solution is obtained.

Seven-Point Formulation

In this section, attention is directed to the application of the above results to the more commonly applied 7-point formulation. This situation can be derived as a special case of the 19-point formulation and requires many fewer algebraic operations for its implementation. In the 7-point formulation, it is noted that

$$A^{bs}_{i,j,k}=A^{bw}_{i,j,k}=A^{be}_{i,j,k}=A^{bn}_{i,j,k}=A^{sw}_{i,j,k}=A^{se}_{i,j,k}$$
$$=A^{nw}_{i,j,k}=A^{ne}_{i,j,k}=A^{fs}_{i,j,k}=A^{fw}_{i,j,k}=A^{fe}_{i,j,k}=A^{fn}_{i,j,k}=0 \qquad (28)$$

which leads to the further results that

$$a_{i,j,k}=b_{i,j,k}=f_{i,j,k}=r_{i,j,k}=v_{i,j,k}=w_{i,j,k}=0 \qquad (29)$$

and that

$$\phi^{1}_{i,j,k}=\phi^{3}_{i,j,k}=\phi^{6}_{i,j,k}=\phi^{9}_{i,j,k}=\phi^{10}_{i,j,k}=\phi^{12}_{i,j,k}$$
$$=\phi^{13}_{i,j,k}=\phi^{15}_{i,j,k}=\phi^{16}_{i,j,k}=\phi^{19}_{i,j,k}=\phi^{22}_{i,j,k}=\phi^{24}_{i,j,k}=0 \qquad (30)$$

Using the above results, the L and U matrices take the form shown in Figs. 5a and 5b, respectively. The modified matrix A' which results from the L and U multiplication is shown in Fig. 6. It is noted that the ϕ are sequentially numbered and do not correspond to those given in the earlier figure. This is due to the drastic reduction in the number of coefficients which result when employing the 7-point formulation. The coefficients of the L and U matrices, including the influence of partial cancellation are given by Eqs. (31-43).

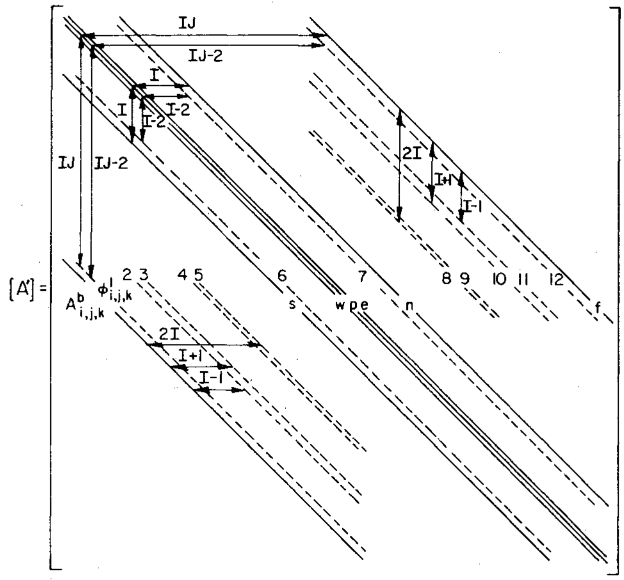


Fig. 6 ***LU* product for 7-point application.**

$$a_{i,j,k}=A^{b}_{i,j,k}/\{1+\alpha[p_{i,j,k-1}-h_{i,j,k-1}(h_{i+1,j,k-1}+r_{i+1,j,k-1})$$
$$-(r_{i,j,k-1}-p_{i+1,j,k-1}h_{i,j,k-1})(h_{i,j+1,k-1}$$
$$+p_{i,j+1,k-1}+r_{i,j+1,k-1})]\} \qquad (31)$$

$$b_{i,j,k}=-a_{i,j,k}h_{i,j,k-1} \qquad (32)$$

$$c_{i,j,k}=-a_{i,j,k}r_{i,j,k-1}-b_{i,j,k}p_{i+1,j,k-1} \qquad (33)$$

$$d_{i,j,k}=\{A^{s}_{i,j,k}-a_{i,j,k}s_{i,j,k-1}+\alpha[(h_{i+1,j-1,k}+2s_{i+1,j-1,k}$$
$$+v_{i+1,j-1,k})b_{i,j,k}s_{i+1,j,k-1}-s_{i-1,j,k}(A^{w}_{i,j,k}$$
$$-a_{i,j,k}u_{i,j,k-1})]\}/\{1+\alpha[2s_{i,j-1,k}+u_{i,j-1,k}$$
$$-s_{i-1,j,k}p_{i,j-1,k}-h_{i,j-1,k}(h_{i+1,j-1,k}$$
$$+2s_{i+1,j-1,k}+v_{i+1,j-1,k})]\} \qquad (34)$$

$$e_{i,j,k}=-b_{i,j,k}s_{i+1,j,k-1}-d_{i,j,k}h_{i,j-1,k} \qquad (35)$$

$$f_{i,j,k}=[A^{w}_{i,j,k}-a_{i,j,k}u_{i,j,k-1}-d_{i,j,k}p_{i,j-1,k}$$
$$-\alpha(a_{i,j,k}p_{i,j,k-1}+c_{i,j,k}p_{i,j+1,k-1}$$
$$+d_{i,j,k}u_{i,j-1,k})]/[1+\alpha(2p_{i-1,j,k}+s_{i-1,j,k}+2u_{i-1,j,k})] \qquad (36)$$

$$g_{i,j,k}=A^{p}_{i,j,k}-a_{i,j,k}v_{i,j,k-1}-b_{i,j,k}u_{i+1,j,k-1}-c_{i,j,k}s_{i,j+1,k-1}$$
$$-d_{i,j,k}r_{i,j-1,k}-e_{i,j,k}p_{i+1,j-1,k}-f_{i,j,k}h_{i-1,j,k}$$
$$+\alpha[2(\phi^{1}_{i,j,k}+\phi^{2}_{i,j,k}+\phi^{3}_{i,j,k})+3\phi^{4}_{i,j,k}$$
$$+2(\phi^{5}_{i,j,k}+\phi^{6}_{i,j,k}+\phi^{7}_{i,j,k}+\phi^{8}_{i,j,k})+3\phi^{9}_{i,j,k}$$
$$+2(\phi^{10}_{i,j,k}+\phi^{11}_{i,j,k}+\phi^{12}_{i,j,k})] \qquad (37)$$

$$h_{i,j,k}=[A^{e}_{i,j,k}-b_{i,j,k}v_{i+1,j,k-1}-e_{i,j,k}r_{i+1,j-1,k}$$
$$-\alpha(2\phi^{1}_{i,j,k}+\phi^{3}_{i,j,k}+2\phi^{6}_{i,j,k}+\phi^{9}_{i,j,k}+\phi^{11}_{i,j,k})]/g_{i,j,k} \qquad (38)$$

$$p_{i,j,k} = (-c_{i,j,k}u_{i,j+1,k-1} - f_{i,j,k}r_{i-1,j,k})/g_{i,j,k} \tag{39}$$

$$r_{i,j,k} = [A^n_{i,j,k} - c_{i,j,k}v_{i,j+1,k-1} - \alpha(\phi^2_{i,j,k} + \phi^3_{i,j,k} + 2\phi^4_{i,j,k} + 2\phi^5_{i,j,k} + \phi^7_{i,j,k})]/g_{i,j,k} \tag{40}$$

$$s_{i,j,k} = (-d_{i,j,k}v_{i,j-1,k} - e_{i,j,k}u_{i+1,j-1,k})/g_{i,j,k} \tag{41}$$

$$u_{i,j,k} = (-f_{i,j,k}v_{i-1,j,k})/g_{i,j,k} \tag{42}$$

$$v_{i,j,k} = [A^f_{i,j,k} - \alpha(\phi^8_{i,j,k} + \phi^9_{i,j,k} + \phi^{10}_{i,j,k} + \phi^{11}_{i,j,k} + \phi^{12}_{i,j,k})]/g_{i,j,k} \tag{43}$$

The additional terms, the ϕ's, are obtained directly from the LU product operations and have the following values:

$$\phi^1_{i,j,k} = b_{i,j,k}h_{i+1,j,k-1} \tag{44}$$

$$\phi^2_{i,j,k} = a_{i,j,k}p_{i,j,k-1} \tag{45}$$

$$\phi^3_{i,j,k} = b_{i,j,k}r_{i+1,j,k-1} + c_{i,j,k}h_{i,j+1,k-1} \tag{46}$$

$$\phi^4_{i,j,k} = c_{i,j,k}p_{i,j,k-1} \tag{47}$$

$$\phi^5_{i,j,k} = c_{i,j,k}r_{i,j+1,k-1} \tag{48}$$

$$\phi^6_{i,j,k} = e_{i,j,k}h_{i+1,j-1,k} \tag{49}$$

$$\phi^7_{i,j,k} = f_{i,j,k}p_{i-1,j,k} \tag{50}$$

$$\phi^8_{i,j,k} = d_{i,j,k}s_{i,j-1,k} \tag{51}$$

$$\phi^9_{i,j,k} = e_{i,j,k}s_{i+1,j-1,k} \tag{52}$$

$$\phi^{10}_{i,j,k} = d_{i,j,k}u_{i,j-1,k} + f_{i,j,k}s_{i-1,j,k} \tag{53}$$

$$\phi^{11}_{i,j,k} = e_{i,j,k}v_{i+1,j-1,k} \tag{54}$$

$$\phi^{12}_{i,j,k} = f_{i,j,k}u_{i-1,j,k} \tag{55}$$

Equations (31-55) give the coefficients of the L and U matrices and of the additional coefficients. It is noted that, since certain of the LU coefficients involve the additional coefficients, the ϕ's should be evaluated as soon as f is known. The numerical molecule representation of A' is presented in Fig. 7. Note that although there are more additional coefficients than in the SIP procedure, the positions of the coefficients in this MSI procedure are further removed from the central point than are those of the SIP procedure and their influence is therefore expected to be weaker.

Iterative Procedure

Following the same procedure outlined by Stone for the SIP procedure,[2] we form the iterative equation

$$[A+B]\{T\}^{n+1} = [A+B]\{T\}^n - ([A]\{T\}^n - \{q\}) \tag{56}$$

Defining a difference vector and a residual vector according to the relations

$$\{\delta\}^{n+1} \equiv \{T\}^{n+1} - \{T\}^n \tag{57}$$

$$\{R\}^n \equiv \{q\} - [A]\{T\}^n \tag{58}$$

the iteration equation, Eq. (56), becomes

$$[A+B]\{\delta\}^{n+1} = \{R\}^n \tag{59}$$

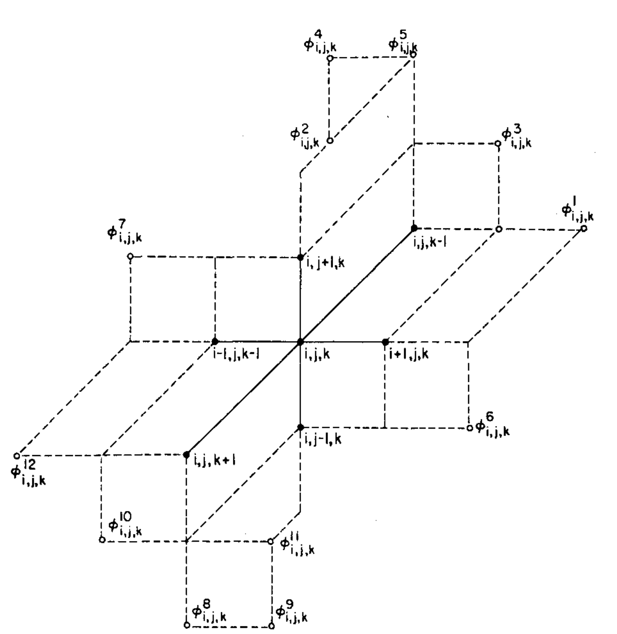


Fig. 7 Numerical molecule representation of 7-point formulation.

wherein the elements of B are the ϕ's as given above. Replacing $[A+B]$ by the LU product results in

$$[L][U]\{\delta\}^{n+1} = \{R\}^n \tag{60}$$

Defining an intermediate vector V by

$$\{V\}^{n+1} \equiv [U]\{\delta\}^{n+1} \tag{61}$$

The solution procedure takes on the familiar two-step process given by

$$[L]\{V\}^{n+1} = \{R\}^n \tag{62}$$

$$[U]\{\delta\}^{n+1} = \{V\}^{n+1} \tag{63}$$

The process represented by Eqs. (62) and (63) consists simply of a forward substitution followed by a backward substitution. Since the coefficients remain unchanged for the process, each iteration requires simply an evaluation of the new residual vector followed by a forward and backward triangular substitution process.

Testing and Application

In the previous sections of this paper, the formulation of the modified strongly implicit procedure for three-dimensional conduction heat transfer has been presented. While the MSI equations appear to have a more complicated form than those of other available procedures, namely ADI and SIP, the programming of these equations is not difficult. Indeed, all coefficients for the forward and backward elimination steps can be explicitly evaluated. The final step in the presentation of a new procedure is to evaluate its performance with respect to that of the alternative procedures.

The MSI procedure has been extensively examined to determine its sensitivity to both problem and procedure parameters. A comprehensive presentation of these sensitivities is available in Ref. 11 but, due to space limitations, only a representative presentation will be provided here. Following the discussion of the testing of the method to specific model problems, a demonstration of the application of the MSI procedure to a more complex physical problem will be provided.

The sensitivity analysis of the MSI procedure was performed for a three-dimensional formulation of a one-dimensional problem with heat flow in each of the three possible coordinate system orientations and for various values of grid aspect ratio. These model problems are defined in Fig. 8 and are denoted by the symbolism MODXX and XX is a two-digit code used to define the problem as indicated in the figure. Not all MODs will be explicitly discussed, again due to space limitations.

The effect of the partial cancellation factor α on the MSI performance for heat flow in the x direction is illustrated in Fig. 9 for the $7\times7\times7$ grid used in the study. It is observed that, for different numbers of iterations, the optimum value of α is relatively high. The influence of α on accuracy increases significantly as larger numbers of iterations are employed. For aspect ratios, a/c from Fig. 8, greater than 5, the optimum value of α is unity. The influence of aspect ratio is indicated in Fig. 9c. These experiments indicate that, where the heat flow is in the x direction, the larger the stretching of the control volume along this direction, the larger will be both the range and the optimum value of α. For stretching of the control volume in either direction normal to the x direction, not explicitly presented here, the range of α remains the same although the optimum value of α decreases to a value near 0.5.

An indication of the sensitivity of the MSI procedure to heat flow direction is provided in Fig. 10 where a comparison is also provided with the SIP and ADI methods. It is observed that the characteristic curves of the MSI procedure for the three different heat flow directions are very similar while the range of α is different for each case. For heat flow in the z direction, for example, the MSI range of α is restricted to values less than 0.4. However, for the same computational time t_f the MSI procedure provides an average error of at least two orders of magnitude less than that of SIP. Figure 10b presents a comparison between the MSI, SIP, and ADI methods for the three MOD involved. Note that for each method, a complete study was performed to determine the optimum solution parameters for the particular problem. It is seen that the MSI procedure provides the highest convergence rate for the three test problems. The SIP method requires a computational time of at least 1.8 times that of the MSI procedure to achieve a desired accuracy. The advantage of MSI over SIP is more pronounced for higher accuracy requirements. The range of α for the MSI procedure depends on the main direction of heat flow, but the characteristic curves are similar for all three directions. In all cases, the MSI procedure offers the most economical results. It is recommended, however, that wherever possible the x coordinate direction be aligned with the predominant heat flow direction.

In the opposite extreme to the results of Fig. 10, in a performance sense, are the results corresponding to a grid aspect ratio of 10. These results are presented in Fig. 11. It is seen, in this extreme aspect ratio situation, that the range of α is considerably reduced for the cases in which the heat flow direction is in the y or z coordinate directions. In all cases,

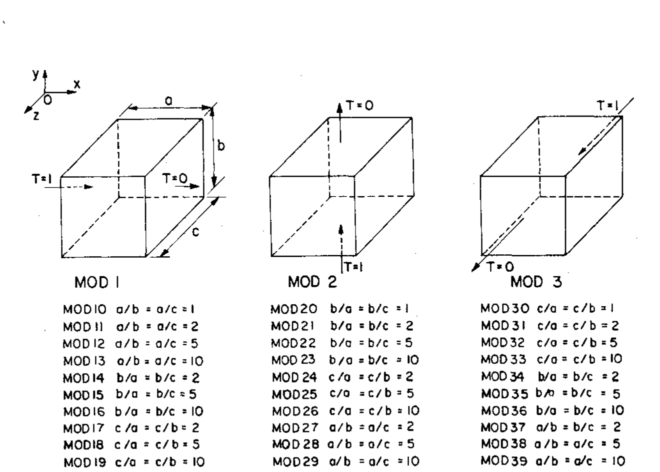


Fig. 8 Model problems for testing of MSI procedure.

however, the MSI performance remains considerably better than the SIP performance, although the ADI method provides comparable results when the heat flow is in the y or z coordinate directions. For the heat flow in the x direction, the MSI performance is far superior to that of the other methods. Although the aspect ratio of 10 represents an extreme case, not frequently encountered in applications, these results emphasize the desirability of aligning the predominant heat flow direction with the x coordinate direction.

For aspect ratios less than unity, i.e., stretching the control volumes in a direction normal to the heat flow direction, the results are more favorable to the MSI procedure than those presented above. Experiments indicate that the effect of stretching the control volume in either of the two directions

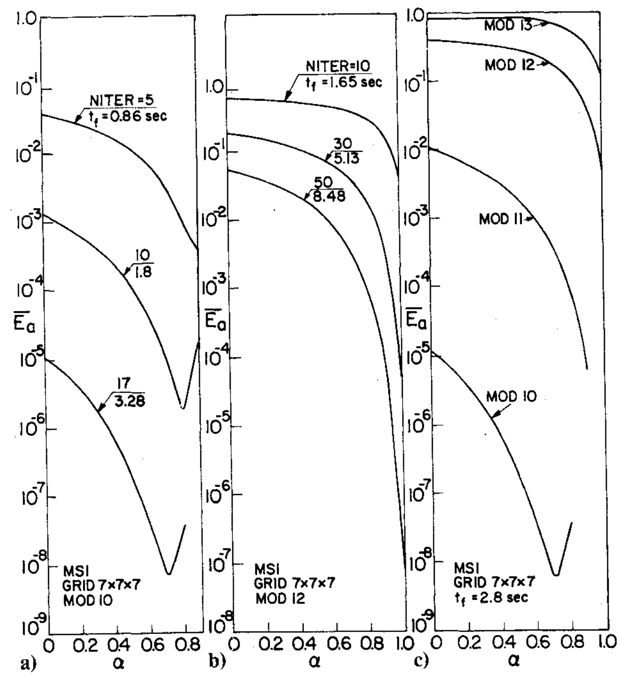


Fig. 9 Influence of α on MSI convergence.

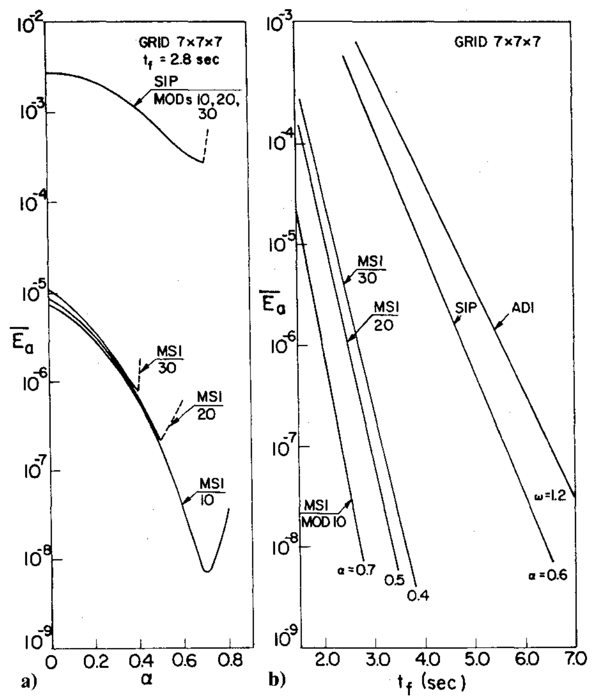


Fig. 10 Comparison of MSI, SIP, and ADI methods for selected model problems.

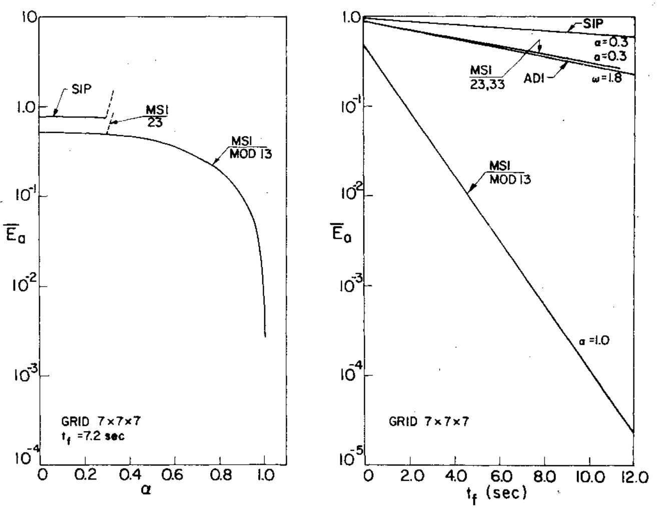


Fig. 11 Comparison of MSI, SIP, and ADI methods for aspect ratio of 10.

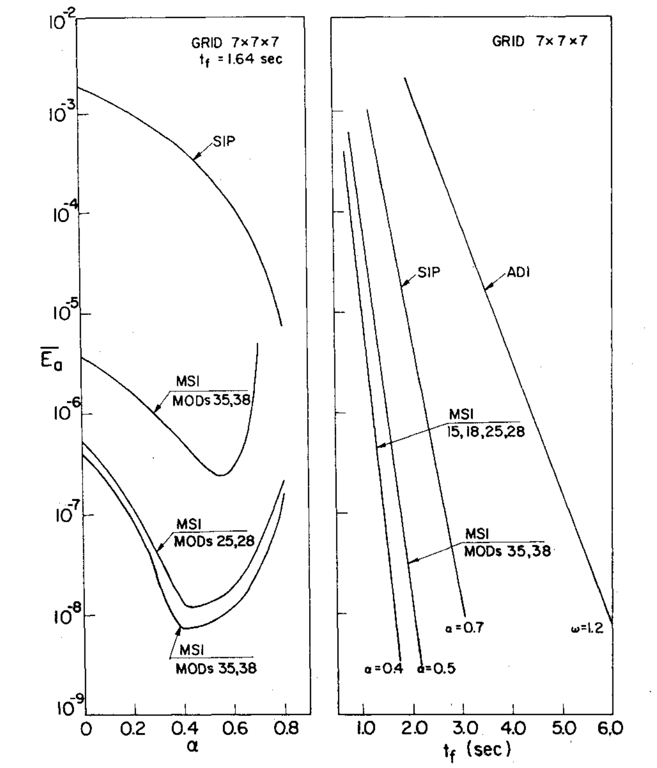


Fig. 12 Comparison of MSI, SIP, and ADI methods for aspect ratio of 0.2.

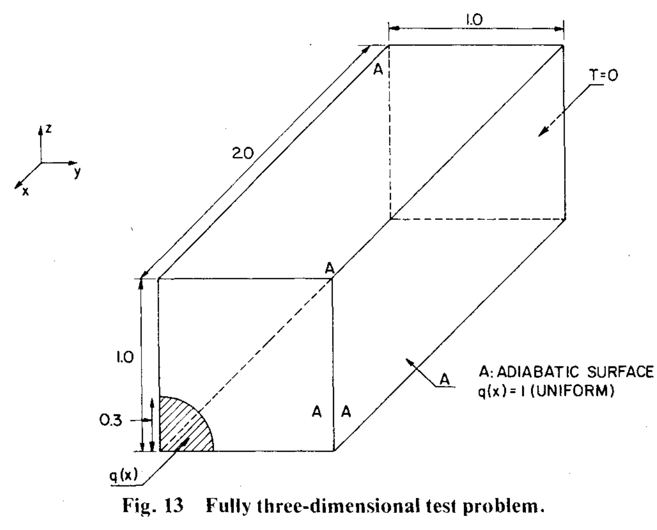


Fig. 13 Fully three-dimensional test problem.

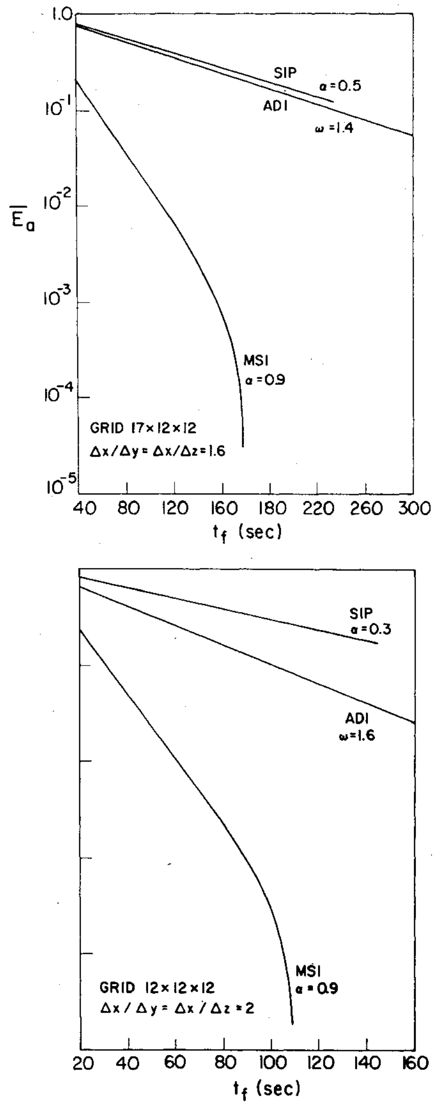


Fig. 14 Comparison of MSI, SIP, and ADI methods for fully three-dimensional test problem.

normal to the heat flow direction yields essentially identical results. Typical results, for an aspect ratio of five, are presented in Fig. 12. It is observed that the range of α for the MSI method is approximately the same for the three directions with the optimum value being near 0.4. It is seen from the figure that the SIP method provides faster convergence than does the ADI method, but that the MSI method offers convergence characteristics superior to both the SIP and ADI results.

Finally, as a demonstration of the procedure to a more demanding heat flow problem, the MSI procedure has been applied to a problem involving steady-state heat flow, with no heat generation, in the three dimensions of a Cartesian coordinate system. The particular problem examined is illustrated in Fig. 13 where a uniform heat flux is prescribed over the shaded area of the figure. The opposite, outflow boundary is specified to be isothermal at $T=0$ and the remaining surfaces are adiabatic. A convergence study for this problem indicated a smooth asymptotic convergence towards the analytical solution given by Beck[12] as the number of control volumes was increased. The error in the numerical solution was less than 5% for a grid $25\times17\times17$ in the x, y, and z directions, respectively. Specific comparisons of the computational times for the MSI procedure with the SIP and ADI methods are provided in Fig. 14 for grids of $17\times12\times12$ and $12\times12\times12$. It is noted that the x dimension is twice the y and z dimensions so that the above grids represent aspect ratios of 1.5 and 2.0, respectively. It is noted that the ADI method provides more accurate results than does the SIP method for both cases. Moreover, it is further noted that the MSI procedure provides significantly better performance than either the ADI method or the SIP method. Specifically, for both mesh sizes used, the MSI procedure provides a factor of five savings in computational time if only a 10% error in the discrete solution is desired. For higher accuracy requirements the savings increase further as is evident from the characteristic curves shown in Fig. 14.

Discussion and Conclusions

A modified strongly implicit procedure for the solution of three-dimensional heat conduction problems has been presented. This work extends the previous work of Schneider and Zedan[6] which was directed at two-dimensional applications. The method presented in this work is formulated as a 19-point scheme, in three dimensions, and the more common 7-point scheme is extracted as a special case of the 19-point scheme. The motivation has been to reduce the influence of asymmetry which results from the additional terms in the LU product, to remove the requirement of equation reordering after every iteration and to reduce the sensitivity of the procedure to problem and procedure parameters.

The new procedure has been extensively examined through its application to the test problems defined in Fig. 8. Through its application to these relatively simple problems, it was possible to examine in detail the attributes of the method and to make comparisons with the strongly implicit procedure and the ADI procedure. Having established the characteristics of the method, it was possible to examine a more complex, fully three-dimensional problem. The method performed well on all of these problems, with the largest computational savings realized for the more complex, fully three-dimensional problem.

On the basis of the testing and experience gained with the new MSI method for three dimensions, and the comparisons made with the alternative procedures, the following conclusions can be advanced regarding the procedure:

1) The MSI procedure is less sensitive to the partial cancellation parameter α than is the SIP procedure.

2) Renumbering of the grid network after every iteration is not required.

3) The MSI procedure can be employed for a very wide range of $\Delta x/\Delta y$.

4) The computational savings range from a factor of unity to an order of magnitude.

In view of the above conclusions, and their relevance to the state-of-the-art of solving field problems, we feel that this work provides a significant contribution to those involved in the solution of field problems. Methods for increasing the range of α, particularly where the predominant heat flow direction is in the z coordinate direction, are currently being investigated.

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